MONOHARMONICALLY DRIVEN NONLINEAR ELASTICALLY HEREDITARY OSCILLATOR

Yu. I. Karkovskii and S. I. Meshkov UDC 539.376+539.135

An analysis of the steady-state forced oscillatons of an elastically hereditary single-mass system with three terms included in the Volterra–Fréchet multiple-integral relations [1] makes it possible to determine the fundamental laws governing the frequency dependence of the amplitude and phase [2]. Also of interest is the analogous problem with all terms included in the multiple-integral relations. The authors realize this possibility in the general case. Separable weighting functions consisting of products of exponential relaxation kernels with a separated instantaneous part are considered as a concrete example.

1. For a one-dimensional oscillator driven by an external monoharmonic force the equation of motion in the coordinate is written in the form

$$M\ddot{x}+f(x,\ \dot{x})=b\cos\omega t, \tag{1.1}$$

where M is the mass, $f(x, \dot{x})$ is the restoring force of the system, b is the amplitude of the driving force, ω is the cyclic frequency, and t is the time.

According to the equivalent linearization method [3], Eq. (1.1) can be rewritten

$$M\ddot{x}+\omega^{-1}\eta\dot{x}+kx+\varepsilon(x,\ \dot{x})=b\cos\omega t. \tag{1.2}$$

Here $\varepsilon(x, \dot{x})$ denotes the error due to replacement of the nonlinear function $f(x, \dot{x})$ by the equivalent linear viscoelastic part:

$$\varepsilon(x,\ \dot{x})=f(x,\ \dot{x})-kx-\omega^{-1}\eta\dot{x}. \tag{1.3}$$

The steady-state solution of Eq. (1.2) for $\varepsilon(x, \dot{x}) = 0$

$$x=a\cos\theta,\ \theta=\omega t-\varphi \tag{1.4}$$

enables us to find the amplitude a and the tangent of the phase shift angle:

$$a=b\,[\eta^2+(k-M\omega^2)^2]^{-\frac{1}{2}};\quad \mathrm{tg}\,\varphi=\eta\,(k-M\omega^2)^{-1}. \tag{1.5}$$

The coefficients k and η are evaluated from the condition for minimum error $\varepsilon(x, \dot{x})$, as expressed by the following two equations averaged over the oscillation period [4]:

$$\left\langle\frac{\partial}{\partial k}[\varepsilon(x,\dot{x})]^2\right\rangle=0;\quad \left\langle\frac{\partial}{\partial\eta}[\varepsilon(x,\dot{x})]^2\right\rangle=0. \tag{1.6}$$

Kuibyshev. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 185-190, July-August, 1975. Original article submitted October 10, 1974.

This optimality criterion for k and η is best [5] when the nonlinear equation (1.1) is replaced by its linearized equivalent (1.2).

The substitution of (1.3) and (1.4) into (1.6) yields expressions accounting for the nonlinear properties of the system:

$$k = \frac{1}{\pi a}\int_0^{2\pi} f(a,\theta)\cos\theta d\theta, \quad \eta = -\frac{1}{\pi a}\int_0^{2\pi} f(a,\theta)\sin\theta d\theta. \tag{1.7}$$

The quantity k plays the part of the dynamic modulus, and η is proportional to the area of the hysteresis loop. As a measure of the internal friction we determine the reciprocal quality factor Q^{-1}:

$$Q^{-1} = \frac{\Delta W}{2\pi W} = \frac{2}{\pi k a^2}\int_0^T \dot{x}\cos\omega t dt = \frac{b\sin\varphi}{ak} = \frac{\eta}{k}.$$

In calculating the elastic strain energy W here we use the dynamic modulus. The reciprocal quality Q^{-1} coincides with the tangent of the phase shift angle, tan φ in the quasistatic case, i.e., for M = 0.

2. The given analytical method can be applied to the elastically hereditary system

$$f(x,\dot{x}) = \sum_{n=1}^{\infty}\int_0^{\infty}\overset{n}{\dots}\int g_n(t_1,\dots,t_n)\prod_{i=1}^{n} x(t-t_i)\,dt_i. \tag{2.1}$$

By substituting (2.1) into (1.7) with regard for (1.4) we can find the equivalent linearization coefficients.

Restricting the problem to separable [6] weighting functions for simplicity,

$$g_n(t_1,\dots,t_n) = \prod_{i=1}^{n} g(t_i); \quad g(t) = E_\infty[\delta(t) - \nu_\varepsilon R(t)], \qquad \nu_\varepsilon \equiv (E_\infty - E_0)E_\infty^{-1}, \tag{2.2}$$

where R(t) is the relaxation kernel, δ(t) is the delta function, and E_∞, E_0 are the nonrelaxed and relaxed values of the elastic modulus, we obtain

$$k = g'(\omega)F\left(1,\frac{3}{2};\ 2;\ 1-y^2\right) = g'(\omega)\frac{2}{y(1+y)}; \qquad \eta = -g''(\omega)\frac{2}{y(1+y)}; \quad y \equiv \sqrt{1-a^2|g(\omega)|^2}. \tag{2.3}$$

Here $F(1, 3/2; 2; 1-y^2)$ is a hypergeometric function [7], and

$$g(\omega) \equiv \int_0^{\infty} g(t)\exp(-i\omega t)\,dt; \qquad g'(\omega) \equiv \mathrm{Re}\,g(\omega),\ g''(\omega) \equiv -\mathrm{Im}\,g(\omega). \tag{2.4}$$

From the first relation (1.5) we obtain an expression for the amplitude:

$$P(y) \equiv \Omega^4 y^5 + \Omega^4 y^4 + (b^2C - 4A\Omega^2 - \Omega^4)y^3 + (b^2C - \Omega^4)y^2 + 4(C + A\Omega^2)y - 4C = 0, \tag{2.5}$$

in which the following notation is introduced:

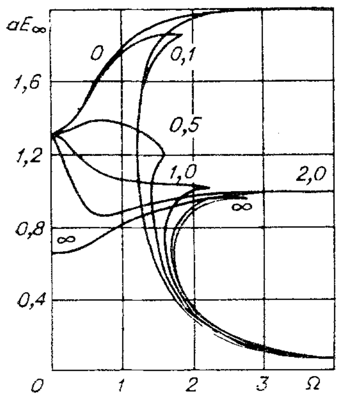

Fig. 1

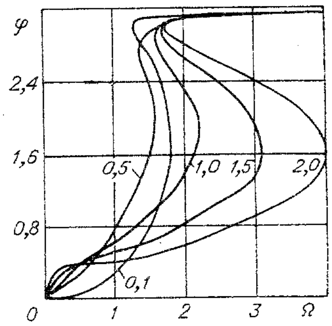

Fig. 2

$$A \equiv E_\infty^{-1} g'(\omega), \quad C \equiv E_\infty^{-2} |g(\omega)|^2; \quad \Omega \equiv \omega\omega_\infty^{-1}, \quad \omega_\infty^2 \equiv E_\infty M^{-1}. \tag{2.6}$$

Solving Eq. (2.5) for y, we find the amplitude and tangent of the phase shift angle:

$$a = E_\infty^{-1} \sqrt{(1-y^2) C^{-1}}; \quad \operatorname{tg} \varphi = -\frac{E_\infty^{-1} g''(\omega)}{A - \Omega^2 (y^2 + y) 2^{-1}}. \tag{2.7}$$

By the obvious inequalities P(0) = $-4C < 0$, P(1) = $2b^2C > 0$ and the occurrence of one or three sign changes [8] in the system for coefficients of the polynomial P(y) for any values of b and Ω it can be verified that Eq. (2.5) has at least one and not more than three roots in the interval (0.1). Consequently, the amplitude and phase determined according to (2.7) are multiple-valued. This property is typical of nonlinear systems [9].

As a concrete example it is instructive to analyze the relxation kernel for a standard linear body:

$$R(t) = \tau_\varepsilon^{-1} \exp(-t\tau_\varepsilon^{-1}), \tag{2.8}$$

since for

$$f(x, \dot{x}) = \int_0^\infty g(t') F[x(t-t')]\, dt',$$

where

$$F[x(t)] = x(t),$$

the steady-state amplitudes have a common point of intersection determined by the "quasiresonance" frequency [10].

Thus, setting the derivative of the amplitude with respect to the relaxation time equal to zero, we obtain the value of the frequency from (1.5):

$$\omega_*^2 = \frac{1}{2M} \frac{\frac{\partial}{\partial \tau_\varepsilon}(k^2 + \eta^2)}{\frac{\partial}{\partial \tau_\varepsilon} k} = \psi(a) \frac{\omega_\infty^2 + \omega_0^2}{2}. \tag{2.9}$$

Here

$$\psi(a) \equiv \frac{1}{\pi a} \int_0^{2\pi} F(a \cos\theta) \cos\theta\, d\theta, \quad \omega_0^2 \equiv E_0 M^{-1}.$$

The coefficients k and η are evaluated from Eqs. (1.7).

Substituting the values of k and η from Eqs. (2.3) into (2.9) with regard for (2.2), (2.4), and (2.8), we obtain the expression

$$\omega_*^2 = \omega_\infty^2 \frac{1+y(1-y)}{y^3(1+y)}\left[\frac{2^1}{2-\nu_\varepsilon} + \frac{(1-y)(1+2y)}{2y^2}\,\frac{1-\nu_\varepsilon+\omega^2\tau_\varepsilon^2}{(1-\nu_\varepsilon)^2+\omega^2\tau_\varepsilon^2}\right]^{-1},$$

which depends on the relaxation time τ_ε. Therefore, a common point of intersection of the amplitudes does not exist except for a particular form of nonlinearity, which in a special case is equivalent to the assumption of similarity of the isochromic fatigue curves [11], in which case the quantity x(t) has the sense of a displacement. The frequency dependence of the amplitude and phase shift angle is given in Figs. 1 and 2, respectively, for the following numerical values of the parameters: b = 1, ν_ε = 0.5. The relaxation times are indicated alongside the curves. The upper branch of the curve for $\tau_\varepsilon = \infty$ in Fig. 1 tends asymptotically to a constant value E_∞^{-1} as $\Omega \to \infty$ and is plotted only in the frequency interval $\Omega \in [0, 2.5]$ to avoid overlapping with the curve for τ_ε = 2, the upper branch of which is terminated at $\Omega \approx 4$.

It is evident in Fig. 1 that a common point of intersection of the amplitude does not exist and the amplitudes remain finite for any values of the frequency, including both associated elastic cases ($\tau_\varepsilon = 0$, $\tau_\varepsilon = \infty$).

In these extreme cases, to obtain the amplitude as a function of the frequency it is convenient to use the inverse dependence of the frequency on the amplitude without resorting to the solution of Eq. (2.5). Solving Eq. (1.5) for the mass-reactance term, we obtain

$$\text{for}\quad \tau_\varepsilon = 0,\quad \omega^2 = \omega_0^2\left[F\left(1, \frac{3}{2}; 2; a^2E_0^2\right) \pm \frac{b}{aE_0}\right];$$
$$\tau_\varepsilon = \infty,\quad \omega^2 = \omega_\infty^2\left[F\left(1, \frac{3}{2}; 2; a^2E_\infty^2\right) \pm \frac{b}{aE_\infty}\right].$$

Hence we infer that the amplitude–frequency curves are obtainable one from the other by a change of scale along the a and ω axes. As $aE_0 \to 1$, $aE_\infty \to 1$, and $a \to 0$ the values of ω^2 grow indefinitely large. Consequently, as $\Omega \to \infty$ Eq. (2.5) always has three roots,

$$1 > y_3 > y_2 > y_1 > 0;\; y^2 \equiv \begin{Bmatrix}1-aE_0\\ 1-aE_\infty\end{Bmatrix}. \tag{2.10}$$

The following limit relations are valid here:

$$\lim_{\Omega\to\infty}(y_1, y_2) = 0;\quad \lim_{\Omega\to\infty} y_3 = 1. \tag{2.11}$$

It can be verified on the basis of expressions (2.10) and (2.11) with allowance for the last expression (2.3) that for the associated elastic cases there are three values of the amplitude for any sufficiently large frequencies, where

$$0 < a_3 < a_2 < a_1 < \begin{cases}E_0^{-1}, & \tau_\varepsilon = 0\\ E_\infty^{-1}, & \tau_\varepsilon = \infty\end{cases}$$

and

$$\lim_{\omega\to\infty}(a_1, a_2) = \begin{cases}E_0^{-1}, & \tau_\varepsilon = 0\\ E_\infty^{-1}, & \tau_\varepsilon = \infty\end{cases},\qquad \lim_{\omega\to\infty} a_3 = 0.$$

Inasmuch as for small values of Ω the system of coefficients of Eq. (2.5) has only one sign change, there is only one positive root in this case. The frequency above which Eq. (2.5) has three positive roots is given by the relation

$$\Omega_+^2 = F\left(1, \frac{3}{2}; 2; z_*\right) + \frac{b}{z_*}, \quad \Omega_+^2 \equiv \begin{Bmatrix} \omega^2\omega_0^{-2} \\ \omega^2\omega_\infty^{-2} \end{Bmatrix}.$$

Here z_* is the root of the equation

$$\frac{d}{dz}\left[F\left(1, \frac{3}{2}; 2; z\right) + \frac{b}{z}\right] = 0,$$

which is taken by the change of variable

$$z = 2\xi(1+\xi^2)^{-1}$$

into the form

$$\xi^5 + \xi^3 + 12^{-1}\nu\xi^2 - 12^{-1}\nu = 0.$$

For all intermediate relaxation times $0 < \tau_\varepsilon < \infty$ the amplitudes and phases are calculated according to Eqs. (2.7). Here the values of y belonging to the interval (0.1) are determined from Eq. (2.5), the coefficients of which are calculated for each frequency according to Eqs. (2.6) with regard for (2.2), (2.4) and (2.8).

Thus, the analysis of the steady-state forced vibrations of a nonlinear elastically hereditary oscillator by the equivalent linearization method makes it possible to explain the principal characteristics of the frequency dependence of the amplitudes and phases of the vibrations for different values of the rheological parameters. We have shown that even for exponential relaxation kernels a common point of intersection of amplitudes does not exist as in the linear case [12] or as in the case of a special type of nonlinearity [10] such that the integral operator represents a Hammerstein operator [13], i.e., the product of a linear operator and a nonlinear superposition operator.

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